On spatially-growing finite disturbances in plane Poiseuille flow

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Possible solutions of the Navier–Stokes equations are given representing certain finite disturbances in plane Poiseuille flow which vary with *distance* parallel to the bounding walls. These solutions are based on infinitesimal disturbances which vary exponentially with *distance* (upstream or downstream) instead of with time, and they are more closely related to the disturbances investigated experimentally than the corresponding 'time-dependent' solutions.

1. Introduction

Contributions to the non-linear theory of the mechanics of instability in either parallel flow or thermal convection have been given by, for example, Landau (1944), Meksyn & Stuart (1951), Gorkov (1957), Malkus & Veronis (1958), Stuart (1956*a*, *b*, 1958, 1960, 1961*b*), Veronis (1959), Benney & Lin (1960) and Watson (1960). For a detailed account of these and other works the reader is referred to Stuart (1961*a*).

In nearly all of the above papers the non-linear theory is based upon the standard linear theory, a detailed treatment of which is given by Lin (1955). The theoretical investigation into the stability of a given parallel (or nearly parallel) flow is undertaken on the basis of a linear theory in which the disturbance takes the form of a wave travelling with constant velocity and with an amplitude which is the product of a function of the distance normal to the boundary with an exponential function of time. The homogeneous equation and boundary conditions represent an eigenvalue problem for determining this wave velocity and the rate of amplification in terms of the wave-number (α) of the disturbance and the Reynolds number (R) of the flow. In general a number of eigenvalues is theoretically possible for a given wave-number and given Reynolds number. For many of the basic flows considered (including plane Poiseuille flow), within a certain range of variation of wave-number and Reynolds number, it is found that, for given α and R, one, and probably only one, of these disturbances amplifies with time (supercritical disturbances), while outside this region all such disturbances decay (subcritical disturbances). We are therefore presented with the problem of retaining the full equations of motion and trying (*a*) to trace the growth of supercritical disturbances, possibly to a state of equilibrium, and (*b*) to trace subcritical disturbances backwards in time, possibly to some initial state of equilibrium. For the case of plane Poiseuille flow Stuart (1960) overcame an essential difficulty in these problems and showed how to obtain the most important terms in possible solutions to one or the other (but not both) of these problems for points in the (α, R)-plane sufficiently close to the neutral curve. By a reformulation of the problem Watson (1960) was able to show how the full solutions could be obtained.

These non-linear solutions represent disturbances which are periodic in distance parallel to the bounding walls and grow exponentially with *time*; on the other hand, the disturbances investigated experimentally are quasi-steady and vary in amplitude with *distance downstream*. An improved model is therefore obtained in this paper by finding the solutions based on a linear theory in which the disturbance is again a wave travelling with constant velocity but with an amplitude which depends exponentially on *distance downstream* (instead of on time). Mathematically this merely means that the roles of time and distance downstream and the roles of wave-number and frequency are interchanged; the characteristics of this linear theory parallel those for the standard theory and, for example, the characteristics of the neutral disturbances of one linear theory follow from the characteristics of the neutral disturbances of the other linear theory. Hence the methods used by Stuart (1960) and Watson (1960) may be used here to obtain possible solutions. Only the most likely possibility is treated and only the most important terms in the solutions are dealt with. It is pointed out that in experiments the disturbances are introduced at a finite point and the solutions obtained will not represent the experimental disturbances near this point; it might be expected, however, that the solutions represent disturbances which approximate to the experimental disturbances sufficiently far downstream of this point.

Experimental disturbances usually become three-dimensional downstream of the point at which the disturbances are introduced. This could be taken into account by considering three-dimensional disturbances in place of two-dimensional ones (cf. Stuart 1961 *b*; Benney & Lin 1960).

For convenience we refer to the problem in which the amplitude of the disturbance varies with time as the 'time-dependent' case and to the problem in which the amplitudes varies with distance downstream as the 'spatially-dependent' case.

2. Linear theory

In two-dimensional, incompressible flow between parallel planes let x denote the distance parallel to the planes, z the distance normal to them measured from the channel centre, ψ the stream function, R the Reynolds number and t the time. Then the vorticity equation may be written in the form

$$\zeta_t + \psi_z \zeta_x - \psi_x \zeta_z = R^{-1}\nabla^2\zeta, \tag{2.1}$$

where
$$\zeta = \nabla^2\psi = \psi_{xx} + \psi_{zz}, \tag{2.2}$$

and the suffices indicate differentiation. All quantities have been made non-dimensional, the reference length (h) being half the distance between the planes, the reference velocity (U_0) being the maximum velocity in steady flow and the reference time being h/U_0. Then the Reynolds number is $R = U_0 h/\nu$, where ν is the kinematic viscosity. The basic steady flow considered is plane Poiseuille flow, for which

$$\psi = \int_0^z (1-z^2)\,dz. \tag{2.3}$$

The other basic flow is plane Couette flow. This will not be considered here although a formal analysis can be made in a manner similar to that for Poiseuille flow (see Watson 1960).

In investigating the stability of Poiseuille flow the type of infinitesimal disturbance considered is a wave travelling in the direction of flow having a stream function of the form

$$\psi = C\psi_1(z)\,e^{i\alpha(x-ct)} + \tilde{C}\tilde{\psi}_1(z)\,e^{-i\alpha(x-\tilde{c}t)}, \tag{2.4}$$

where α is real [from the symmetry of (2.4) it is evident that there will be no loss in generality if α is assumed to be positive], the symbol $\sim$ denotes a complex conjugate, and C is an arbitrary constant. The linear equation for $\psi_1(z)$ is the Orr–Sommerfeld equation,

$$(\bar{u}_l - c)(\psi_1'' - \alpha^2\psi_1) - \bar{u}_l''\psi_1 + (i/\alpha R)(\psi_1^{\text{iv}} - 2\alpha^2\psi_1'' + \alpha^4\psi_1) = 0, \tag{2.5}$$

where an accent (superscript) indicates differentiation with respect to z and where $\bar{u}_l = 1-z^2$. This equation, together with the homogeneous boundary conditions corresponding to the vanishing of the velocity components on the planes, constitutes an eigenvalue problem to determine $c = c_r + ic_i$ as a function of α and R (c will in general be complex). There are two fundamental sets of solutions, the set of those solutions which are even functions of z and the set of solutions which are odd functions of z; any solution is a linear combination of a solution from each set. Now from physical considerations the given flow will be stable if R is sufficiently small. Instability (corresponding to eigenvalues with $c_i > 0$) has been found only for the set of solutions even in z and, corresponding to any given point (α, R) in the unstable region of the (α, R)-plane, only one eigenvalue with $c_i > 0$ has been found (it has also been found that c_r is positive for this disturbance). Hence the instability characteristics can be found by considering the even solutions of (2.5). Since the equation and boundary conditions are homogeneous, ψ_1 is determined apart from an arbitrary multiplicative factor. Since C is arbitrary, ψ_1 can be made definite by imposing the normalizing condition $\psi_1(0) = 1$. The neutral curve in the (α, R)-plane separating stable from unstable disturbances is determined by those solutions for which $c_i = 0$. This neutral curve has been found (see Lin 1955). Having selected the desired eigenvalue and eigenfunction, Stuart (1960) and Watson (1960) have used (2.4) as a basis for a time-dependent non-linear theory.

This, however, is not a suitable model for the disturbances investigated experimentally, which are quasi-steady and vary in amplitude with distance downstream. A better model is obtained by first introducing the new linear theory below and using this as the basis of a non-linear theory similar to the time-dependent one. The type of disturbance considered here is again one travelling in the direction of flow but having a stream function of the form

$$\psi = C\psi_1(z)\,e^{i(\alpha x-\beta t)} + \tilde{C}\tilde{\psi}_1(z)\,e^{-i(\tilde{\alpha}x-\beta t)}, \tag{2.6}$$

where β is real [from the symmetry of (2.6) there is no loss of generality if β is assumed to be positive] and C is an arbitrary constant.† The linear equation for ψ_1 is

$$L(\alpha,\beta)\,\psi_1 \equiv (\bar{u}_l-\beta/\alpha)\,(\psi_1''-\alpha^2\psi_1)-\bar{u}_l''\,\psi_1+(i/\alpha R)\,(\psi_1^{\mathrm{iv}}-2\alpha^2\psi_1''+\alpha^4\psi_1) = 0, \tag{2.7}$$

and the boundary conditions are the same as in the time-dependent case—the disturbance velocities vanish on the walls so that ψ_1 and ψ_1' vanish at $z = \pm 1$. This eigenvalue problem will give $\alpha = \alpha_r + i\alpha_i$ as a function of β and R (α will be complex in general). The given flow will be unstable to disturbances of the form (2.6) if, for some values of β and R, an eigenvalue α can be found with α_i negative. If α_i is put equal to zero in (2.7) then it is obvious that the required neutral solutions of (2.7) are found immediately from the neutral ($c_i = 0$) solutions of (2.5), the only solutions which are not identically zero being the solutions which are even functions of z. Since C is an arbitrary constant we can make the function ψ_1 definite by imposing the normalizing condition $\psi_1(0) = 1$. Then the neutral solutions of (2.7) will be identical with the neutral solutions of (2.5) (α_r will of course be positive for these disturbances). The neutral curve in the (β, R)-plane for disturbances (2.6) follows immediately from the eigenvalues for the neutral disturbances of (2.4). Since instability can only occur for ψ_1 continued analytically from the neutral solutions into solutions with α complex and β real, ψ_1 remains an even function of z. It might be expected that (i) for any given values of β and R such that the point (β, R) lies inside the neutral curve, we shall find one and probably only one eigenvalue α with α_i negative, in which case (2.6) will be an unstable disturbance, and (ii) for any given β and R such that the point lies outside the neutral curve, we shall find no unstable disturbances. For points near the neutral curve Gaster (1962) has pointed out that the above conjectures are true and that it is probable to prove in this case that the transformation is based on a group velocity. Earlier Schlichting (1933) used the group velocity in this way although apparently without mathematical proof. As in the time-dependent case these two regions in the (β, R)-plane will be called the supercritical and subcritical regions respectively. The stream function (2.6) contains the amplitude factor $e^{-\alpha_i x}$ so that the linear theory can be said to be valid only when $e^{-\alpha_i x}$ is very small. This will be true only far upstream in the supercritical case ($\alpha_i < 0$) and far downstream in the subcritical case ($\alpha_i > 0$).

3. Non-linear theory

The stream function (2.6) involves the sum of terms of the form $f(x,z)\,e^{-i\beta t}$ and $F(x,z)\,e^{i\beta t}$. As the growth of the disturbance is traced, downstream in the supercritical case, upstream in the subcritical case, each of these terms interacts with itself, with the other and with the main flow, modifying them and generating higher harmonics of the form $f_n(x,z)\,e^{ni\beta t}$ ($n = \pm 2, \pm 3, \ldots$). It therefore appears permissible to expand the stream function of the flow when non-linearity is included as a Fourier series in t.

† This type of disturbance has been used, in connection with another problem, by M. Gaster (1962).

Let the stream function for the flow be represented by the Fourier series

$$\psi(x,z,t) = \bar{\phi}+\phi' = \bar{\phi}(x,z)+\sum_{n=1}^{\infty}\{\phi^{(n)}(x,z)\,e^{-ni\beta t}+\tilde{\phi}^{(n)}(x,z)\,e^{ni\beta t}\}. \tag{3.1}$$

As the amplitude of the disturbance tends to zero $\bar{\phi}$ tends to the undisturbed stream function,

$$\int_0^z \bar{u}_l\,dz,$$

and the other part of the right-hand side of (3.1) tends to the linear disturbance stream-function (2.6). When the disturbance has non-zero amplitude the sum on the right-hand side of (3.1) represents the stream function of the disturbance while $\bar{\phi}$ is the mean stream function, where the mean is taken with respect to t over the period of the disturbance, $2\pi/\beta$. The expression (3.1) is to be substituted into (2.1) and the Fourier components are to be equated. The equation arising from equating the terms independent of t is equivalently found by taking the mean of equation (2.1), and this can be written in the form

$$\bar{\phi}_z\nabla^2\bar{\phi}_x-\bar{\phi}_x\nabla^2\bar{\phi}_z+(\overline{\phi'_z\nabla^2\phi'_x-\phi'_x\nabla^2\phi'_z}) = R^{-1}\nabla^4\bar{\phi}. \tag{3.2}$$

The term $(\overline{\phi'_z\nabla^2\phi'_x-\phi'_x\nabla^2\phi'_z})$ is the Reynolds-stress term representing the effect of the disturbance on the mean motion; in the linear theory it is neglected. The disturbance equation is found on subtracting (3.2) from (2.1) and it can be written in the form

$$\nabla^2\phi'_t+\bar{\phi}_z\nabla^2\phi'_x+\phi'_z\nabla^2\bar{\phi}_x-\bar{\phi}_x\nabla^2\phi'_z-\phi'_x\nabla^2\bar{\phi}_z+\chi = R^{-1}\nabla^4\phi', \tag{3.3}$$

where

$$\chi = \phi'_z\nabla^2\phi'_x-\phi'_x\nabla^2\phi'_z-(\overline{\phi'_z\nabla^2\phi'_x-\phi'_x\nabla^2\phi'_z}). \tag{3.4}$$

In the disturbance equations χ is the non-linear part, which is neglected in the linear theory.

Equations (3.2) and (3.3) have to be solved for $\bar{\phi}$ and ϕ'. When the expression for ϕ' [equation (3.1)] is used and components of the Fourier series are separated we obtain the equations

$$\bar{\phi}_z\nabla^2\bar{\phi}_x-\bar{\phi}_x\nabla^2\bar{\phi}_z-R^{-1}\nabla^4\bar{\phi} = \sum_{n=1}^{\infty}(\phi_x^{(n)}\nabla^2\tilde{\phi}_z^{(n)}+\tilde{\phi}_x^{(n)}\nabla^2\phi_z^{(n)}-\phi_z^{(n)}\nabla^2\tilde{\phi}_x^{(n)}-\tilde{\phi}_z^{(n)}\nabla^2\phi_x^{(n)}), \tag{3.5}$$

$$\begin{aligned}\bar{\phi}_z\nabla^2\phi_x^{(n)}-ni\alpha_r c\nabla^2\phi^{(n)}+\phi_z^{(n)}\nabla^2\bar{\phi}_x-\bar{\phi}_x\nabla^2\phi_z^{(n)}-\phi_x^{(n)}\nabla^2\bar{\phi}_z-R^{-1}\nabla^4\phi^{(n)} \\ = -\sum_{m=1}^{n-1}(\phi_z^{(m)}\nabla^2\phi_x^{(n-m)}-\phi_x^{(m)}\nabla^2\phi_z^{(n-m)})-\sum_{m=n+1}^{\infty}(\phi_z^{(m)}\nabla^2\tilde{\phi}_x^{(m-n)}-\phi_x^{(m)}\nabla^2\tilde{\phi}_z^{(m-n)}) \\ -\sum_{m=1}^{\infty}(\tilde{\phi}_z^{(m)}\nabla^2\phi_x^{(n+m)}-\tilde{\phi}_x^{(m)}\nabla^2\phi_z^{(n+m)}) \quad (n \geqslant 1).\end{aligned} \tag{3.6}$$

In (3.6) it is understood that the summations from $m = 1$ to $m = n-1$ are to be omitted when $n = 1$.

The boundary conditions to be imposed are that the velocities $(\psi_z, -\psi_x)$ vanish on the bounding planes, $z = \pm 1$, at all times and for all x. This gives

$$\bar{\phi}_z = \bar{\phi}_x = \phi_z^{(n)} = \phi_x^{(n)} = 0 \quad \text{at} \quad z = \pm 1 \text{ for all } x$$

or, for all x,

$$\bar{\phi}_z = \phi_z^{(n)} = 0,\ \bar{\phi} \text{ and } \phi^{(n)} \text{ const. at } z = \pm 1. \tag{3.7}$$

In addition an 'initial' condition has to be imposed: as the amplitude of the disturbance tends to zero, the disturbance must tend through the infinitesimal disturbance (2.6) to zero (as $x \to \pm\infty$, according to whether the flow is subcritical or supercritical). Since the undisturbed state is attained as $x \to \pm\infty$ it follows that the boundary conditions (3.7) for all x are†

$$\bar{\phi}_z = \phi_z^{(n)} = \phi^{(n)} = 0, \quad \bar{\phi} = \int_0^{\pm 1} \bar{u}_l dz \quad \text{at} \quad z = \pm 1. \tag{3.8}$$

It follows from (3.8) that the volume flux is an absolute constant. Following earlier work (see Watson 1960) we seek a solution of (3.5), (3.6) in the form

$$\phi^{(n)} = A^n\left\{\psi_n + \sum_{m=1}^{\infty} |A|^{2m}\psi_{nm}\right\} \quad (n \geqslant 1), \tag{3.9}$$

$$\bar{\phi} = \int_0^z \bar{u}_l dz + \sum_{m=1}^{\infty} |A|^{2m} f_m, \tag{3.10}$$

and

$$\frac{dA}{dx} = A \sum_{m=0}^{\infty} a_m |A|^{2m} \quad (a_0 = i\alpha), \tag{3.11}$$

where the 'amplitude' A is a function of x only, ψ_n, ψ_{nm}, f_m are functions of z only and a_m ($m \geqslant 1$) are constants to be determined. That $a_0 = i\alpha$ follows from the 'initial' condition. It can readily be shown that the equation for $|A|^2$ is

$$\frac{d|A|^2}{dx} = 2|A|^2 \sum_{m=0}^{\infty} a_{mr} |A|^{2m} \quad (a_m = a_{mr} + ia_{mi}), \tag{3.12}$$

from which it follows that $|A|^2$ is monotonic in x between zero and the first positive zero of the right-hand side of (3.12).

It can be shown that, when (3.9)–(3.11) are substituted into (3.5) and (3.6), an infinite set of equations is obtained in which each term in each equation is the product of a power of $|A|^2$ with a function of z. Since $|A|^2$ varies with x in a range of $|A|^2$ including zero then, in each equation, the coefficients of $|A|^{2m}$ ($m = 1, 2, \ldots$) must cancel. This gives the differential equations for the functions of z in (3.9) and (3.10); these differential equations must be solved in a definite order. Until these functions of z have been calculated it is not known exactly how they behave for small values of α_i. However, let us assume that the most likely behaviour does occur (cf. Watson 1960). Then the terms in (3.9) to (3.11) decrease in magnitude as m increases and the first-order solution is obtained by retaining terms up to order $|A|^3$ and neglecting smaller-order terms. We therefore need to know ψ_{11}, ψ_2, ψ_3, f_1 and a_1 (as well as ψ_1 of course); of these functions ψ_1 must be calculated first and ψ_{11} and a_1 last. The equations for these functions are

$$L(\alpha, \beta)\,\psi_1 = 0, \tag{3.13}$$

$$L(2\alpha, 2\beta)\,\psi_2 = \tfrac{1}{2}(\psi_1\psi_1''' - \psi_1'\psi_1''), \tag{3.14}$$

$$\begin{aligned} L(3\alpha, 3\beta)\,\psi_3 = \tfrac{1}{3}[&\psi_1(\psi_2''' - 4\alpha^2\psi_2') \\ &+ 2\psi_2(\psi_1''' - \alpha^2\psi_1') - 2\psi_1'(\psi_2'' - 4\alpha^2\psi_2) - \psi_2'(\psi_1'' - \alpha^2\psi_1)], \end{aligned} \tag{3.15}$$

$$\begin{aligned} R^{-1}&(f_1^{\text{iv}} + 8\alpha_i^2 f_1'' + 16\alpha_i^4 f_1) + 2\alpha_i\{\bar{u}_l(f_1'' + 4\alpha_i^2 f_1) - \bar{u}_l'' f_1\} \\ &= i\alpha[\tilde{\psi}_1'(\psi_1'' - \alpha^2\psi_1) - \psi_1(\tilde{\psi}_1''' - \tilde{\alpha}^2\tilde{\psi}_1')] - i\tilde{\alpha}[\psi_1'(\tilde{\psi}_1'' - \tilde{\alpha}^2\tilde{\psi}_1) - \tilde{\psi}_1(\psi_1''' - \alpha^2\psi_1')], \end{aligned} \tag{3.16}$$

$$L(\alpha, \beta)\,\psi_{11} - (2i\alpha_i/\alpha)\,g(\psi_{11}) = -(ia_1/\alpha)\,g(\psi_1) + g_{11}, \tag{3.17}$$

† It appears reasonable from physical considerations that the boundary and 'initial' conditions which we have imposed are the correct boundary conditions.

where

$$g(\psi) \equiv \{-\bar{u}_l + [4i(\alpha+i\alpha_i)/R]\}\psi'' + \{\bar{u}_l(3\alpha^2+6i\alpha\alpha_i-4\alpha_i^2)-2\beta(\alpha+i\alpha_i)+\bar{u}_l'' - [4i(\alpha+i\alpha_i)/R](\alpha^2+2i\alpha\alpha_i-2\alpha_i^2)\}\psi$$

and

$$g_{11} \equiv \alpha^{-1}[\alpha\psi_1(f_1''' + 4\alpha_i^2 f_1') - \alpha f_1'(\psi_1''-\alpha^2\psi_1) + 2i\alpha_i f_1(\psi_1'''-\alpha^2\psi_1') - 2i\alpha_i\psi_1'(f_1''+4\alpha_i^2 f_1) + 2\alpha\psi_2(\tilde{\psi}_1'''-\tilde{\alpha}^2\tilde{\psi}_1') - 2\alpha\tilde{\psi}_1'(\psi_2''-4\alpha^2\psi_2) + \tilde{\alpha}\psi_2'(\tilde{\psi}_1''-\tilde{\alpha}^2\tilde{\psi}_1) - \tilde{\alpha}\tilde{\psi}_1(\psi_2'''-4\alpha^2\psi_2')].$$

The boundary conditions corresponding to these equations have still to be given. Since the boundary conditions (3.8) have to be satisfied for all values of x and so for all $|A|$ sufficiently small, then it is easily shown from (3.9) to (3.11) and (3.12) that the boundary conditions are

$$\psi_n = \psi_n' = \psi_{nm} = \psi_{nm}' = f_m = f_m' = 0 \quad \text{at} \quad z = \pm 1. \tag{3.18}$$

Since ψ_1 is an even function of z it is readily seen from (3.14) and (3.18) that ψ_2 is odd, from (3.15) and (3.18) that ψ_3 is even, from (3.16) and (3.18) that f_1 is odd and from (3.17) and (3.18) that ψ_{11} is even. The equations (3.13) to (3.17) need therefore only be solved between $z=0$ and $z=1$, when the boundary conditions are

$$\left.\begin{aligned} \psi_1' = \psi_1''' = \psi_2 = \psi_2'' = \psi_3' = \psi_3''' = f_1 = f_1'' = \psi_{11}' = \psi_{11}''' = 0 \quad &\text{at} \quad z=0,\\ \psi_1 = \psi_1' = \psi_2 = \psi_2' = \psi_3 = \psi_3' = f_1 = f_1' = \psi_{11} = \psi_{11}' = 0 \quad &\text{at} \quad z=1, \end{aligned}\right\} \tag{3.19}$$

together with the normalizing condition $\psi_1(0) = 1$.

Now if we write ψ_{11} in the form

$$\psi_{11} = (a_1/2\alpha_i)\psi_1 + \bar{\psi}_{11}, \tag{3.20}$$

then, from (3.17), $\bar{\psi}_{11}$ satisfies

$$L(\alpha,\beta)\bar{\psi}_{11} - (2i\alpha_i/\alpha)g(\bar{\psi}_{11}) = g_{11} \tag{3.21}$$

and $\bar{\psi}_{11}$ satisfies the same boundary conditions as ψ_{11}. The most likely case, which is considered here, is when $\bar{\psi}_{11}$ is expanded in the series

$$\bar{\psi}_{11} = (1/\alpha_i)\psi_{11}^{(-1)} + \psi_{11}^{(0)} + \alpha_i\psi_{11}^{(1)} + \dots, \tag{3.22}$$

where the functions on the right-hand side of (3.22) are bounded, satisfy

$$L(\alpha,\beta)\psi_{11}^{(-1)} = 0, \tag{3.23}$$

$$L(\alpha,\beta)\psi_{11}^{(0)} = (2i/\alpha)g(\psi_{11}^{(-1)}) + g_{11}, \tag{3.24}$$

$$L(\alpha,\beta)\psi_{11}^{(r)} = (2i/\alpha)g(\psi_{11}^{(r-1)}) \quad (r \geqslant 1), \tag{3.25}$$

respectively, and satisfy the same boundary conditions as ψ_{11}. To the order which is being considered it is necessary to find the required solutions of (3.23) and (3.24) only, for which it is necessary to use (3.25) with $r = 1$. Define χ_3 to be the solution of the Orr–Sommerfeld equation satisfying

$$\chi_3 = \chi_3' = 0, \quad \chi_3'' = 1, \quad \chi_3''' = 0 \quad \text{at} \quad z = 0 \tag{3.26}$$

and Φ by $$\Phi = (\chi_3'' - \alpha^2\chi_3) - \{\chi_3''(1) - \alpha^2\chi_3(1)\}(\psi_1'' - \alpha^2\psi_1)/\psi_1''(1)\dagger \tag{3.27}$$

(which satisfies the equation adjoint to the Orr–Sommerfeld equation and satisfies the same boundary conditions as ψ_1). Then the required solution of (3.23) is (see Stuart 1960 and Watson 1960)

$$\psi_{11}^{(-1)} = \lambda\psi_1, \tag{3.28}$$

where λ is given by

$$\lambda = -\alpha\int_0^1 \Phi g_{11}\,dz \Big/ 2i\int_0^1 \Phi g(\psi_1)\,dz. \tag{3.29}$$

The right-hand side of (3.24) is therefore a known even function and since the solution of (3.24) is to be an even function, it must have the form

$$\psi_{11}^{(0)} = A\psi_1 + B\chi_3 + P, \tag{3.30}$$

where P is any even particular integral of (3.24). Either of the two conditions at $z = 1$ will determine B and the other condition at $z = 1$ will be satisfied. It follows from equation (3.25) with $r = 1$ that the constant A is given by

$$A = -\int_0^1 \Phi g\{B\chi_3 + P\}\,dz \Big/ \int_0^1 \Phi g(\psi_1)\,dz, \tag{3.31}$$

so that both $\psi_{11}^{(-1)}$ and $\psi_{11}^{(0)}$ can be found in this way. The required solutions of (3.25) for any r can be found in a similar manner. It can be shown that a_1 is arbitrary so that in particular it can be chosen to have the value

$$a_1 = -2\lambda, \tag{3.32}$$

when, from (3.20), $$\psi_{11} = \psi_{11}^{(0)} + \alpha_i\psi_{11}^{(1)} + \dots. \tag{3.33}$$

This choice of a_1 makes both a_1 and ψ_{11} bounded, however small α_i may be. To the order considered

$$\psi_{11} = \psi_{11}^{(0)}. \tag{3.34}$$

When $\psi_1, \psi_2, \psi_3, f_1, \psi_{11}^{(0)}$ and a_1 have been found then the solution (3.9), (3.10) will have been found up to terms of order $|A|^3$ after the function A has been calculated from (3.11) retaining only the terms of order $|A|^3$. It will therefore be necessary to solve

$$dA/dx = A(i\alpha + a_1|A|^2), \tag{3.35}$$

from which, of course,

$$d|A|^2/dx = 2|A|^2(-\alpha_i + a_{1r}|A|^2). \tag{3.36}$$

† The analysis in this paper does not hold near points in the (β, R)-plane where there is a complementary function of (3.14) or (3.15) which satisfies the boundary conditions or at points where either $\psi_1''(1) = 0$ or $\psi_1(0) = 0$. At a point where $\psi_1(0) = 0$ the normalizing condition can be taken as $\psi_1''(0) = 1$ when $\psi_1 \equiv \chi_3$, and the analysis remains unaltered if, in the text, χ_3 is replaced by χ_1, the solution of the Orr–Sommerfeld equation satisfying

$$\chi_1 = 1, \quad \chi_1' = \chi_1'' = \chi_1''' = 0 \quad \text{at} \quad z = 0.$$

It readily follows from (3.36) that

$$|A|^2 = -\alpha_i K\, e^{-2\alpha_i x}/(1 - a_{1r} K\, e^{-2\alpha_i x}), \tag{3.37}$$

where K is an arbitrary constant of integration. Therefore as $|A| \to 0$, that is, as $\alpha_i x \to \infty$, $|A|^2 \sim -\alpha_i K\, e^{-2\alpha_i x}$. But since $A \sim C\, e^{i\alpha x}$ as $|A| \to 0$ then we can say that C and K are related by

$$-\alpha_i K = |C|^2. \tag{3.38}$$

The equilibrium amplitude, $|A|_e$ is found (to the first approximation) from (3.37) to be given by

$$|A|_e^2 = \alpha_i/a_{1r}. \tag{3.39}$$

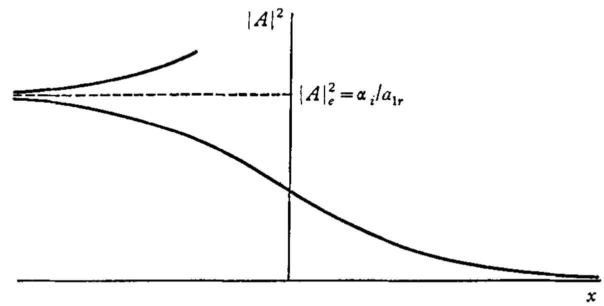

FIGURE 1. Possibility for spatial growth of amplitude in plane Poiseuille flow—subcritical case ($a_{1r} > 0$).

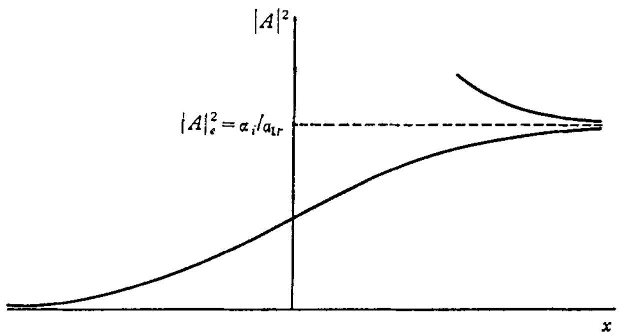

FIGURE 2. Possibility for spatial growth of amplitude in plane Poiseuille flow—supercritical case ($a_{1r} < 0$).

If a_{1r} turns out to be positive then this equilibrium amplitude can be found only in the subcritical case ($\alpha_i > 0$) (for the behaviour of $|A|^2$ with distance, see figure 1), while if a_{1r} is negative the equilibrium amplitude can be found only in the supercritical case ($\alpha_i < 0$) (see figure 2 for the behaviour of $|A|^2$ with distance, but note that, in experiments, disturbances are introduced at a finite value of x). Now K or $|C|^2$ is arbitrary because it corresponds to the arbitrariness in the position of the origin of the x-axis. There will therefore be no loss in generality if we assume that $|C|^2 = |\alpha_i|$ so that from (3.38), $K = -\operatorname{sgn}\alpha_i$ and (3.37) becomes

$$|A|^2 = |\alpha_i|\, e^{-2\alpha_i x}/(1 + a_{1r} \operatorname{sgn}\alpha_i\, e^{-2\alpha_i x}). \tag{3.40}$$

Also the argument of C is arbitrary because it corresponds to the arbitrariness in the position of the origin of time (we have in effect already specified the origin of the x-axis). There will therefore be no loss in generality if we assume that the argument of C is zero. Then from (3.35) it is readily shown that the argument of A follows from

$$A/\tilde{A} = e^{2i\gamma x}, \tag{3.41}$$

where

$$\gamma = \alpha_r - (a_{1i}/2a_{1r}x)\log(1 + a_{1r}\operatorname{sgn}\alpha_i\, e^{-2\alpha_i x}) \tag{3.42}$$

is the wave-number of the disturbance and it changes with distance parallel to the walls. In deriving this result from (3.35) and its complex conjugate, (3.36) and (3.40) have been used. This has the proper behaviour as $A \to 0$; it is easily seen that $\gamma \to \alpha_r$ as $A \to 0$. Also in the equilibrium state

$$\gamma = \alpha_r + (a_{1i}/a_{1r})\,\alpha_i. \tag{3.43}$$

The speed S of the disturbance also varies with distance parallel to the walls and is given by

$$S = \beta/\gamma, \tag{3.44}$$

but the frequency does not change.

When an equilibrium state exists, as in the case we are considering, then the same analysis shows that there is a similar disturbance for $|A| > |A|_e$, although a second equilibrium state probably could not be calculated. If such equilibrium states exist in the subcritical region ($\alpha_i > 0$) then these disturbances amplify with distance downstream, so that the states of equilibrium are unstable, showing that, although the basic flow is stable to infinitesimal disturbances in this region, it can be unstable to certain finite disturbances. On the other hand, if such equilibrium states exist in the supercritical region ($\alpha_i < 0$) then these disturbances decay with distance downstream to the states of equilibrium, so that the states of equilibrium may be stable to infinitesimal disturbances.

In the problems considered by Stuart (1960) and Watson (1960), the flow is periodic in x and the mean is taken with respect to x; if we take the mean of the continuity equation and apply the boundary condition that $\overline{w} = 0$ on the walls, it follows that $\overline{w} \equiv 0$ so that the mean flow remains parallel for all times. Here, however, the mean is taken with respect to t and it does not follow that the mean flow is parallel. In fact, from (3.10), $\overline{w} = -\overline{\phi}_x$ varies with x so that the mean flow is not parallel. If an equilibrium state exists then, as it is approached, $|A|$ will tend to a constant and $\overline{w}$ will tend to zero, that is, the mean flow will become parallel. Another point is that in the time-dependent problem the mean pressure gradient had to be specified to get a unique solution (for example, the mean pressure gradient could be assumed to be an absolute constant or it could be chosen to make the mass-flux constant). In the spatially-dependent problem, however, no corresponding condition need be imposed—the mass flux *is* constant.

By comparing this analysis with the time-dependent growth analysis it can be shown that, in the limit as the neutral curve is approached, the value of a_1 in one case can readily be obtained from the value of a_1 in the other case.

The author is indebted to his colleague, Dr J. T. Stuart, for suggesting this problem, for his many helpful criticisms and advice. The author also wishes to thank Mr M. Gaster for allowing him to study some of his unpublished work.

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